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INTEGRAL SOLUTIONS OF THE WAVE EQUATION AND THE DIFFRACTION OF AN ARBITRARY ACOUSTIC WAVE BY A WEDGE*

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A new solution of the wave equation is obtained: the integral of a product of two functions, the first of which is an arbitrary solution of the equation and the second of which is the derivative of an arbitrary solution of Laplace's equation with respect to the integration parameter. A solution to the problem of an arbitrary wave diffracted by a wedge is determined using the arbitrary wave diffracted by a wedge is determined using the integral of the potential of the wave and the solution of the same problem for a plane unit wave which satisfies Laplace's equation. When the diffraction of waves is considered in the case of spherical symmetry, the resulting integral may be reduced to a known form /1/.

In the case of waves propagating in an angle of half-aperture $\pi/(2n+1)$ and diffracted by a half-plane, the solutions can be expressed in terms of elementary functions.

Since the construction employed here does not differ in its essentials from that described in /1/, the details are omitted.

1. We consider diffraction by a Riemann surface of an arbitrary acoustic wave with potential (or excess pressure beyond the front)

$$f(t, r, z, \cos\theta) H(\eta(t, r, z) - \cos\theta) \tag{1.1}$$

which is a generalized solution of the wave equation

$$\Box f = \frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial z^2} - \frac{\partial^2 f}{\partial t^2} = 0 \tag{1.2}$$

$$r^2 = x^2 + y^2, \ \theta = \operatorname{arctg} y/x$$

Here r, θ, z are cylindrical coordinates and $H(s)$ is Heaviside's function. We shall henceforth refer to a wave with potential (1.1) as a wave of type (1.1). The Heaviside factor in (1.1) indicates that the wave front may be expressed as follows:

$$\eta(t, r, z) - \cos\theta = 0 \tag{1.3}$$

where, since the incident wave front is tangent to the diffracted wave front at $\theta = 0, 2\pi$, i.e., $\cos\theta = 1$, the latter may be expressed as $\eta = 1$.

We seek a solution with periodicity $T = 4\pi - 4\beta$ in θ, which implies that there is an incident wave whenever $(4\pi - 4\beta)k \leqslant \theta \leqslant 2\pi + (4\pi - 4\beta)k$ but no incident wave when $2\pi + (4\pi - 4\beta)k < \theta < (4\pi - 4\beta)(k+1)$, where $k = 0; \pm 1; \pm 2; \ldots$ (shown in the figure for a plane unit wave with $T = 3\pi$). The easiest case to understand is $T = 4\pi$, $\beta = 0$ (corresponding to a half-plane), when the incident wave impinges on the Riemann surface through a sheet (at $4\pi k \leqslant \theta \leqslant 2\pi + 4\pi k$, $k = 0; \pm 1; \pm 2; \ldots$).

The presence of a wave at specific values of θ may be determined by multiplying the potential of the incident wave by $H\{\sin {}^1/_2 \lambda\theta \sin {}^1/_2 \lambda (2\pi - \theta)\}$, where $\lambda = 2\pi/(4\pi - 4\beta)$.

We expand the potential of the incident wave (1.1) in Fourier series with respect to θ, assuming the above periodicity outside the region of diffraction, subsequently moving into the diffraction region. Throughout, the limits of integration in the integral representation of the Fourier coefficients are taken to be eigenfunctions that have no singularities at the edge of the wedge. We then interchange the order of integration and summation (it is not hard to show that this is legitimate). Summing the series obtained under the integral sign, we obtain the solution within the diffraction region:

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$$\Phi(t, r, z, \theta) = f(t, r, z, \cos\theta) H(\sin {}^1\!/_2 \lambda\theta \sin {}^1\!/_2 \lambda (2\pi - \theta)) - \tag{1.4}$$
$$\frac{1}{\pi}\int_{\zeta^\lambda}^{1} f(t, r, z, {}^1\!/_2 (u^{1/\lambda} + u^{-1/\lambda}))(g(u, \theta) + g(u, 2\pi - \theta))\, du$$
$$\zeta = \eta - \sqrt{\eta^2 - 1},\ g(u, \theta) = \sin\lambda\theta/(1 - 2u\cos\lambda\theta + u^2)$$

At $\lambda = {}^1\!/_2$, $T = 4\pi$ (diffraction by a half-plane) this formula may be rewritten in the form

$$\Phi(t, r, z, \theta) = f(t, r, z, \cos\theta) H(\sin {}^1\!/_2 \theta) - \tag{1.5}$$
$$\frac{1}{2\pi}\int_{-\mu}^{\mu} f(t, r, z, 2v^2 + 1)\frac{\sin {}^1\!/_2\theta\, dv}{v^2 + \sin^2 {}^1\!/_2 \theta}, \quad \mu = \sqrt{\frac{\eta - 1}{2}}$$

2. The same result may be obtained by a different argument. Let us seek a solution of the wave Eq.(1.2) in the form

$$\Phi(t, r, z, \theta) = \int f(t, r, z, \eta)\varphi(\eta, \theta) d\eta$$

where the first function in the integrand is a wave of type (1.1) with front (1.3) and $\varphi(\eta, \theta)$ is a function whose form is yet to be determined.

Since the surface $\eta = \cos\theta$ is a wave front (i.e., a characteristic of Eq.(1.2)), the function $f(t, r, z, \cos\theta)$ may behave there in any of three different ways. In the first case the function is discontinuous across the front (and since $f \equiv 0$ outside the front, the jump is $f(t, r, z, \eta)$). In the second the derivative of the function with respect to the surface (1.3) experiences a discontinuity. Finally, higher-order derivatives of $f(t, r, z, \cos\theta)$ may be discontinuous.

Simple differentiation shows that

$$\Box\Phi = \int \varphi \Box_\eta f\, d\eta + \int f \triangle_\eta \varphi\, d\eta + \varphi \begin{cases} \mathrm{Tr}_\eta (f(t, r, z, \eta)) & \text{(case 1)} \\ \mathrm{Pr}_\eta (f(t, r, z, \eta)) & \text{(case 2)} \\ 0 & \text{(case 3)} \end{cases}$$

Here $\Box\Phi$ is the wave operator (see (1.2)), $\Box_u f$ is the wave operator after substitution of $u = \cos\theta$, and therefore $\Box_\eta f = 0$. In addition, $\mathrm{Tr}_\eta (f(t, r, z, \eta)) = 0$ is the equation governing the propagation of discontinuities of the solutions of Eq.(1.2) /3/, which is satisfied in the first case by $f(t, r, z, \eta)$. The function $\mathrm{Pr}_\eta (f(t, r, z, \eta))$ is the transversal derivative with respect to the surface (1.3). However, this surface is a characteristic, and therefore the transversal direction is tangential /3/. The tangential derivative cannot be discontinuous across a characteristic, but outside the front $f \equiv 0$, and therefore $\mathrm{Pr}_\eta (f(t, r, z, \eta)) = 0$. Thus, for $\Box\Phi$ to vanish we must put

$$\Delta_\eta \varphi = (\eta^2 - 1)\frac{\partial^2\varphi}{\partial\eta^2} + 3\eta\frac{\partial\varphi}{\partial\eta} + \varphi + \frac{\partial^2\varphi}{\partial\theta^2} = 0 \tag{2.1}$$

If we put $\psi = \int \varphi d\eta$, integrate Eq.(2.1) with respect to η and then apply the Chaplygin transformation $\zeta = \eta - \sqrt{\eta^2 - 1}$, the result is the Laplace equation

$$\zeta\frac{\partial}{\partial\zeta}\left(\zeta\frac{\partial\psi}{\partial\zeta}\right) + \frac{\partial^2\psi}{\partial\theta^2} = 0 \tag{2.2}$$

Thus, for any solution ψ of the Laplace Eq.(2.2) and any solution of Eq.(1.2) of type (1.1) with front $\eta = \cos\theta$, we can construct a new solution of the wave equation

$$\Phi(t, r, z, \theta) = \int f(t, r, z, \eta)\frac{\partial}{\partial\eta}\psi(\eta - \sqrt{\eta^2 - 1}, \theta)\, d\eta \tag{2.3}$$

Conversely, if Φ is a solution of the wave equation and we are given an arbitrary wave of type (1.1), then $\psi(\zeta, \theta)$ satisfies Laplace's Eq.(2.2).

Choosing ψ as a plane unit wave satisfying Laplace's equation and diffracted by a wedge, we obtain a new solution of the wave equation, and the form of the integrand in (2.3) is just (1.4).

Using (1.5), it is easy to obtain all the solutions of the problem of waves diffracted by a Riemann surface with periodicity $T = 4\pi$, which were described in /2/ in terms of elementary functions.

We will now consider diffraction by a Riemann surface with periodicity $T = 4\pi$ of a spherical wave

$$f(t, r, z, \cos\theta) = (\tau - \rho)^k H(\tau - \rho)/\rho$$

where $\tau = R_0 + t$ is the time, measured from the formation of the wave, $t = 0$ is the time of arrival of the wave at the $r = 0$ axis along which the Riemann surface branches, $R_0 = \mathrm{const}$ is the distance from the centre of symmetry of the wave to the axis $r = 0$, $\rho^2 = R_0^2 + r^2 + 2R_0 r\cos\theta$, the integral for this function in (1.5) can be evaluated in terms of elementary functions and within the diffraction region one has

$$\Phi = (4R_0r)^{(k-1)/2}\Bigg\{\sum_{n=0}^{[k/2-1/2]}\binom{k}{2n+1}\beta^{k/2-1/2-n}\Bigg[\varepsilon^n\left(\frac{1}{2}+\frac{1}{\pi}\operatorname{arctg}\frac{\psi}{\sqrt{\eta_1}}\right) - \tag{2.4}$$

$$\frac{1}{\pi}\sum_{l=1}^{n}\binom{n}{l}\varepsilon^{n-l}\sum_{m=0}^{l-1}\binom{l}{m}\frac{\eta_1^{m+1/2}}{2m+1}\psi^{2l-2m-1}\Bigg] +$$

$$\sum_{n=0}^{[k/2]}\binom{k}{2n}\beta^{k/2-n}\Bigg\{\varepsilon^{n-1/2}\left(\frac{1}{2}+\frac{1}{\pi}\operatorname{arctg}\frac{\sqrt{\beta}\,\psi}{\sqrt{\eta_1\varepsilon}}\right) -$$

$$\frac{1}{2\pi}\sum_{l=1}^{n}\binom{n}{l}\varepsilon^{n-l}\Bigg\{\beta^{1/2}\sum_{m=1}^{l-1}\binom{l-1}{m}\frac{\psi^{2l-2m-1}}{m}\Bigg[\eta_1^{m-1/2} +$$

$$\sum_{j=1}^{m}\frac{(-1)^j(2m-1)\ldots(2m-2j+1)}{2^j(m-1)\ldots(m-j)}\xi_1^{\,j}\eta_1^{m-j-1/2}\Bigg] +$$

$$\sum_{m=0}^{l-1}\binom{l-1}{m}(-1)^m\psi^{2l-2m-1}\xi_1^{\,m}\frac{(2m-1)!!}{(2m)!!}\ln\frac{\sqrt{\beta}+\sqrt{\eta_1}}{\sqrt{\beta}-\sqrt{\eta_1}}\Bigg\}\Bigg\}\Bigg\}$$

$$\xi_1 = \frac{\xi+1}{2} = \frac{(R_0+r)^2+z^2}{4R_0r},\quad \eta_1 = \frac{\eta-1}{2} = \frac{(R_0+t)^2-(R_0+r)^2-z^2}{4R_0r}$$

$$\psi = \sin{}^1\!/_2\theta,\quad \beta = \xi_1+\eta_1,\quad \varepsilon = \xi_1 - \sin^2{}^1\!/_2\theta$$

Here $\binom{n}{m}$ are the binomial coefficients and $[k/2]$ the integral part of the bracketed number.

3. We will now consider an arbitrary incident wave of the form $f(t, r, z, \theta)H(\eta - \cos\theta)$ with front $\eta = \cos\theta$. Repeating the reasoning of Sect.1, we obtain the solution of the problem of diffraction of the wave by a Riemann surface with periodicity $T = 4\pi - 4\beta$, $\lambda = 2\pi/(4\pi - 4\beta)$:

$$\Phi(t, r, z, \theta) = f(t, r, z, \theta)H(\sin{}^1\!/_2\lambda\theta\sin{}^1\!/_2\lambda(2\pi-\theta)) - \tag{3.1}$$

$$\frac{1}{2\pi}\int_{\zeta^\lambda}^{1}[f(t, r, z, -i\ln u^{1/\lambda}) + f(t, r, z, i\ln u^{1/\lambda})][g(u, \theta) + g(u, 2\pi-\theta)]du +$$

$$\frac{1}{2\pi i}\int_{\zeta^\lambda}^{1}[f(t, r, z, -i\ln u^{1/\lambda}) - f(t, r, z, i\ln u^{1/\lambda})][\chi(u, \theta) + \chi(u, 2\pi-\theta)]du$$

$$i^2 = -1,\quad \chi(u, \theta) = (\cos\lambda\theta - u)/(1 - 2u\cos\lambda\theta + u^2)$$

Here ζ and $g(u, \theta)$ are defined as in (1.4). If $T = 4\pi$, $\lambda = {}^1\!/_2$, formula (3.1) becomes

$$\Phi(t, r, z, \theta) = f(t, r, z, \theta)H(\sin{}^1\!/_2\theta) - \frac{1}{4\pi}\int_{-\mu}^{\mu}[f(-i\ln\sqrt{v^2+1} - v)^2) + \tag{3.2}$$

$$f(i\ln(\sqrt{v^2+1} - v)^2)]\frac{\sin{}^1\!/_2\theta\,dv}{v^2+\sin^2{}^1\!/_2\theta} + \frac{1}{2\pi i}\int_{\nu}^{1}[f(-i\ln(w-\sqrt{w^2-1})^2) -$$

$$f(i\ln(w-\sqrt{w^2-1})^2)]\frac{\cos{}^1\!/_2\theta\,dw}{w^2-\cos^2{}^1\!/_2\theta}$$

$$i^2 = -1,\quad \mu = \sqrt{(\eta-1)/2},\quad \nu = \sqrt{(\eta+1)/2}$$

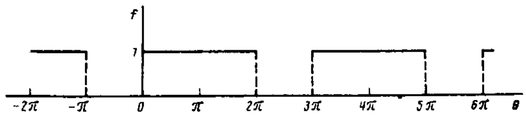
We have omitted the first three arguments in the expression for $f(t, r, z, u)$ in the integrand.

If $f(t, r, z, u)$ is an analytical function of its last argument, expressions (3.1), (3.2) will be real. Otherwise, only the real part of the expression on the right is retained.

4. We will now consider the solution with periodicity $T = 2\pi q/p$, $\lambda = p/q$, where p and q are integers. Analysis shows that one can first find the solution for $\lambda = 1/q$, $T = 2\pi q$, subsequently constructing the solution for $\lambda = p/q$ as

$$f_{p/q}(\theta) = \sum_{k=0}^{p-1} f_{1/q}\left(\theta + \frac{k2\pi q}{p}\right) \tag{4.1}$$

As an illustration, consider the example $\lambda = {}^2/_3$, $T = 3\pi$ and a plane unit wave. In that case a wave exists for $0 \leqslant \theta \leqslant 2\pi,\ 3\pi \leqslant \theta \leqslant 5\pi$ etc., but not for $2\pi < \theta < 3\pi,\ 5\pi < \theta < 6\pi$ (see the figure).



The problem may clearly be split into two with periodicity $T = 6\pi$ ($\lambda = {}^1/_3$). In the first we assume that there is a wave when $6\pi k \leqslant \theta \leqslant 2\pi + 6\pi k$, in the second the non-zero wave exists at $3\pi + 6\pi k \leqslant \theta \leqslant 5\pi + 6\pi k$ but not at $5\pi + 6\pi k < \theta < 8\pi + 6\pi k$. Here, again, $k = 0; \pm 1; \pm 2; \ldots$.

Note that by taking $\theta_1 = \theta - 3\pi$ in the second problem, we are back in the situation of the first problem. It will therefore suffice to determine a solution $f_{1/3}(\theta)$ for the first problem, and then $f_{1/3}(\theta + 3\pi)$ will be a solution of the second. Since the wave Eq.(1.2) is linear and the sum of conditions of the first and second problems satisfies the conditions for $T = 3\pi$, $\lambda = {}^2/_3$, it follows that

$$f_{2/3}(\theta) = f_{1/3}(\theta) + f_{1/3}(\theta) + 3\pi$$

Similar arguments can be used for arbitrary integers p and q and any form of $f(\theta)$. The same result is obtained if the functions $g(u, \theta)$ and $\chi(u, \theta)$ occurring in (1.4) and (3.1) are expanded in fractions corresponding to $\lambda = 1/q$.

If $\lambda = {}^1/_2$, $T = 4\pi$ solutions of the diffraction problem have been determined for some waves in terms of elementary functions. Therefore, using (4.1), one can construct a solution for $\lambda = n + {}^1/_2$, $T = 4\pi/(2n + 1)$ (corresponding to the incidence of the wave in an angular region with half-aperture $\pi/(2n + 1)$):

$$f_{n+1/2}(\theta) = \sum_{k=0}^{2n} f_{1/2}(\theta_k) \tag{4.2}$$

Here and below, $\theta_k = \theta + 4\pi k/(2n + 1)$.

For example, for a cylindrical wave $\ln(\delta - \sqrt{\delta^2 - 1})$, where $\delta^2(\theta) = (R_0 + t)^2/(R_0^2 + r^2 + 2R_0 r\cos\theta)$, $R_0 = \text{const}$ is the distance from the wave axis to the $r = 0$ axis, we obtain

$$f_{n+1/2}(\theta) = \frac{1}{2}\sum_{k=0}^{2n}\left[\ln\left(\delta(\theta_k) - \sqrt{\delta^2(\theta_k) - 1}\right) + \frac{1}{2}\ln\frac{R_0 + r - \sqrt{2R_0 r}\sin{}^1/_2\theta_k}{R_0 + r + \sqrt{2R_0 r}\sin{}^1/_2\theta_k}\right] \tag{4.3}$$

This series may be continued for any solution of the diffraction problem for $T = 4\pi$ which is known in terms of elementary functions.

5. Using formulae (1.4), (1.5), one can investigate the diffraction of a wave $f(t, r, z, \cos(\alpha + \theta))H(\eta - \cos(\alpha + \theta))$ by a wedge and a half-plane, respectively. When constructing the solution it should be remembered that the reflected wave also participates in the diffraction.

If the boundary condition at the wedge surface is $\partial\Phi/\partial n = 0$, where n is the normal to the surface, the reflected wave can be written in the form (1.1) with θ replaced by $2\beta + \alpha - \theta$. If $\Phi = 0$ the reflected wave will have the same form at the wedge surface, but with the opposite sign.

To obtain the solution inside the diffraction region, we must replace θ in formulae (1.4) and (1.5) (or in the particular relations (2.4), (4.1)-(4.3)) by $\alpha + \theta$ (for the incident wave), then replace θ by $2\beta + \alpha - \theta$ (for the reflected wave), and add the results in the case $\partial\Phi/\partial n = 0$ or subtract the first from the second if $\Phi = 0$, i.e., within the diffraction region the result is

$$\Phi = \Phi(\alpha + \theta) \pm \Phi(2\beta + \alpha - \theta)$$

where $\Phi(\theta)$ is the corresponding solution for the Riemann surface.

These constructions using formulae (3.1), (3.2) may be carried out to obtain the solution of the diffraction problem for a wave $f(t, r, z, \alpha + \theta) H(\eta - \cos(\alpha + \theta))$ by a wedge and a half-plane.

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WEAKLY LINEAR OSCILLATIONS OF THE RADIUS OF A VAPOUR BUBBLE IN AN ACOUSTIC FIELD*

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Non-linear heat-and-mass exchange effects between a vapour bubble and a surrounding liquid under periodic pressure oscillations generated by an acoustic field of length significantly greater than the radius of the bubble are investigated. Based on a closed system of equations for the spherically symmetric processes around an isolated bubble /1/, the method of multiple scales /2, 3/ is used to derive asymptotic equations for the behaviour of the average bubble radius, accurate to the second order in the field amplitude.

Linear and weakly linear oscillations of vapour bubbles in acoustic fields have been studied quite extensively, and the main results have been summarized in the literature /1, 4/. The most comprehensive investigation of the "smoothed heat transfer effect" for vapour bubbles, that is, the variation of the average bubble radius over a large number of periods due to the non-linearity of heat-and-mass exchange, may be found in /4/. This paper departs from previous publications on "smoothed heat transfer" in its systematic allowance for the non-equilibrium conditions of the phase transitions, which, over a certain parameter range, exert a decisive effect on the dynamics of the average bubble radius; the non-uniform vapour temperature in the bubble is also taken into account. In addition, application of the method of multiple scales has justified certain assumptions previously adopted in applications of the averaging method to derive equations for the dynamics of the average bubble radius.

1. Statement of the problem. We shall study the behaviour of a spherical vapour bubble in an unbounded space occupied by an ideal incompressible liquid, with the pressure at infinity p_∞ varying periodically about an equilibrium value $p_* = p_s(T_*)$, $T_* = T_\infty$ (T is the temperature, the subscript s denotes the parameters on the saturation curve and the asterisk denotes the parameters of the unperturbed state):

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